

# COMPACT SYMMETRIC SOLUTIONS TO THE POSTAGE STAMP PROBLEM

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**ABSTRACT.** We derive lower and upper bounds on possible growth rates of certain sets of positive integers  $A_k = \{1 = a_1 < a_2 < \dots < a_k\}$  such that all integers  $n \in \{0, 1, 2, \dots, ka_k\}$  can be represented as a sum of no more than  $k$  elements of  $A_k$ , with repetition.

## 1. INTRODUCTION

The postage stamp problem [2, C 12] is a classic problem in additive number theory and can be described as follows: if  $h$  and  $k$  are positive integers,  $A_k = \{1 = a_1 < a_2 < \dots < a_k\}$ ,  $a_i \in \mathbb{N}$  and

$$S(h, A_k) = \left\{ \sum_{i=1}^k x_i a_i \mid x_i \geq 0, \sum_{i=1}^k x_i \leq h \right\}$$

then what is the smallest positive integer  $N(h, A_k) \notin S(h, A_k)$ ? One focus is to solve the global aspect of this problem, that is, given  $h$  and  $k$  find  $A_k$  such that  $N(h, A_k)$  is as large as possible. The case  $k = 3$  was solved by Hofmeister [3], and for  $k \geq 4$  Rødseth [5] derived the currently best known general upper bound. Another focus is to solve the local aspect, that is, given  $h, k$  and  $A_k$  determine  $N(h, A_k)$ . The case  $k = 3$  is covered in [6]. Both aspects were solved for the case  $k = 2$  in [7].

It is easy to see that  $N(h, A_k) \leq ha_k + 1$ . In this paper, we focus on integrating the global and local aspects by investigating certain sets generated for which this inequality is actually an equality.

**1.1. Preliminaries.** From here on we restrict our attention to the situation  $h = k$ . We say a set  $A_k = \{1 = a_1 < a_2 < \dots < a_k\}$  is *symmetric* if when  $k = 2m$  then

$$\begin{aligned} a_1 &= 1 \\ a_i &> a_{i-1} \text{ for } 2 \leq i \leq m \\ a_{m+i} &= 2a_m - a_{m-i} \text{ for } 1 \leq i \leq m-1 \\ a_{2m} &= 2a_m \end{aligned}$$

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and when  $k = 2m + 1$  then

$$\begin{aligned}
a_1 &= 1 \\
a_i &> a_{i-1} \text{ for } 2 \leq i \leq m \\
\hat{a}_m &= a_m + x \text{ for } 0 < x \in \mathbb{N} \\
a_{m+i} &= \hat{a}_m + a_m - a_{m-i} \text{ for } 1 \leq i \leq m-1 \\
a_{2m} &= \hat{a}_m + a_m,
\end{aligned}$$

where the  $2m + 1$  elements are ordered

$$a_1 < a_2 < \dots < a_m < \hat{a}_m < a_{m+1} < \dots < a_{2m}.$$

This labelling of the elements has been chosen to make the enclosed proofs more uniform.

The largest integer that can be represented as the sum of  $k$  positive integers chosen from  $A_k$ , with repetitions allowed, is clearly  $ka_k$ . If every positive integer  $n$ ,  $0 \leq n \leq ka_k$ , can be represented as the sum of at most  $k$  positive integers from  $A_k$ , then we say that  $A_k$  is *compact*. We now study the growth rate of the  $a_i$  such that  $A_k$  is both symmetric and compact. More precisely, if  $A_k = \{1 = a_1 < a_2 < \dots < a_{2m}\}$  is symmetric then we derive bounds  $\alpha, \beta$  such that if  $\frac{a_i}{a_{i-1}} \leq \alpha$  for  $2 \leq i \leq m$  then  $A_k$  will always be compact, whereas if  $\beta \leq \frac{a_i}{a_{i-1}}$  for  $2 \leq i \leq m$  then  $A_k$  will never be compact. Symmetric compact sets were studied in [8] where the focus was on sets with a stronger symmetry property, known as nested symmetry.

For convenience, we refer to  $A_k$  as the *base*, refer to the  $a_i$  as *base elements*, and denote the largest base element by  $M$ .

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## 2. A LOWER BOUND

We now describe symmetric sets  $A_k$  that are compact. For the remainder of this section, let  $A_k = \{1 = a_1 < a_2 < \dots < a_{2m}\}$  be a symmetric base such that

(1) the  $a_i$  satisfy

$$\begin{aligned}
a_1 &= 1 \\
a_i &\leq 3a_{i-1} \text{ for } 2 \leq i \leq m \\
a_{m+i} &= 2a_m - a_{m-i} \text{ for } 1 \leq i \leq m-1 \\
a_{2m} &= 2a_m
\end{aligned}$$

or

(2) the  $a_i$  satisfy

$$\begin{aligned} a_1 &= 1 \\ a_i &\leq 3a_{i-1} \text{ for } 2 \leq i \leq m \\ \hat{a}_m &= a_m + x \text{ for } 0 < x \leq 2a_m \\ a_{m+i} &= \hat{a}_m + a_m - a_{m-i} \text{ for } 1 \leq i \leq m-1 \\ a_{2m} &= \hat{a}_m + a_m. \end{aligned}$$

The following theorem on  $A_k$  can be proved via [4, Korollar], however, we provide a direct proof, which begins with

**Lemma 2.1.** *Let  $1 \leq r \leq m-1$ . If  $\lfloor \frac{n}{M} \rfloor \leq r$  and  $n - \lfloor \frac{n}{M} \rfloor M < a_{r+1}$  then  $n$  can be written as a sum of at most  $2r$  base elements with repetition.*

*Proof.* We proceed by induction on  $r$ . When  $r = 1$  observe that  $n - \lfloor \frac{n}{M} \rfloor M < a_2 \leq 3$ , so  $n = 0, 1, 2, M, M+1, M+2$ . That  $n$  can be written as a sum of two base elements is trivial for all cases bar  $n = M+2$ . This case only arises if  $a_2 = 3$ , in which case we can write  $n = (M-1) + a_2$ .

Now let  $i = \lfloor \frac{n}{M} \rfloor, j = \lfloor \frac{n-iM}{a_r} \rfloor$ . Since  $n - iM < a_{r+1} \leq 3a_r$  we know  $0 \leq j \leq 2$ . If

- (1)  $j \leq 1$  let  $n' = n - ja_r - \min(i, 1)M$
- (2)  $j = 2, i = 0$  let  $n' = n - 2a_r$
- (3)  $j = 2, i > 0$  let  $n' = n - (M - a_r) - a_{r+1}$ .

Note in each of these cases, respectively,  $n' \geq 0$  since

- (1) if  $j = 0$  then  $n - iM \geq 0$ , whereas if  $j = 1$  then  $n - iM \geq a_r$
- (2) if  $j = 2$  and  $i = 0$ , then  $\lfloor \frac{n}{a_r} \rfloor = 2$ , so  $n \geq 2a_r$
- (3) if  $j = 2$  and  $i > 0$ , then  $\lfloor \frac{n-iM}{a_r} \rfloor = 2$ , so  $n - iM \geq 2a_r$  and  $n - iM + a_r - a_{r+1} \geq 3a_r - a_{r+1} \geq 0$ .

Moreover, in each of these cases  $n'$  respectively satisfies

- (1)  $i' = \lfloor \frac{n'}{M} \rfloor = 0$  if  $i = 0$ , or  $i' = \lfloor \frac{n'}{M} \rfloor = i-1 \leq r-1$  otherwise, and  $\lfloor \frac{n'-i'M}{a_r} \rfloor = \lfloor \frac{n-ja_r}{a_r} \rfloor = 0$  if  $i = 0$ , or  $\lfloor \frac{n'-i'M}{a_r} \rfloor = \lfloor \frac{n'-(i-1)M}{a_r} \rfloor = \lfloor \frac{n-ja_r-iM}{a_r} \rfloor = 0$  otherwise, so  $n' - \lfloor \frac{n'}{M} \rfloor M < a_r$
- (2)  $i' = \lfloor \frac{n'}{M} \rfloor = 0 < r-1$  since  $i = 0$  and  $r \geq 2$ , and  $\lfloor \frac{n'-i'M}{a_r} \rfloor = \lfloor \frac{n-2a_r}{a_r} \rfloor = 0$  since  $j = 2$ , so  $n' - \lfloor \frac{n'}{M} \rfloor M < a_r$
- (3)  $i' = \lfloor \frac{n'}{M} \rfloor = \lfloor \frac{n+a_r-a_{r+1}-M}{M} \rfloor = i-1 \leq r-1$  since  $a_r - a_{r+1} < 0$ , and  $\lfloor \frac{n'-i'M}{a_r} \rfloor = \lfloor \frac{n'-(i-1)M}{a_r} \rfloor = \lfloor \frac{n+a_r-a_{r+1}-iM}{a_r} \rfloor = 0$  since  $n - iM < a_{r+1}$ , so  $n - iM - a_{r+1} + a_r < a_r$ , and hence  $n' - \lfloor \frac{n'}{M} \rfloor M < a_r$ .

Thus in each case we can apply the induction hypothesis to  $n'$  and write  $n'$  as a sum of at most  $2(r-1)$  base elements with repetition. The result now follows for  $n$ .  $\square$

**Theorem 2.2.**  $A_k$  is compact.

*Proof. Case (1):* Suppose first that  $n < 2ma_m$ . Consider  $\lfloor \frac{n}{a_m} \rfloor$ . If  $\lfloor \frac{n}{a_m} \rfloor$  is even then  $\lfloor \frac{n}{a_m} \rfloor = 2l \leq 2(m-1)$  and so  $\lfloor \frac{n}{2a_m} \rfloor = \lfloor \frac{n}{M} \rfloor \leq m-1$  and  $n - \lfloor \frac{n}{M} \rfloor M = n - \lfloor \frac{n}{2a_m} \rfloor M < a_m$ . Thus, by Lemma 2.1  $n$  can be written as a sum of at most  $2(m-1)$  base elements. If  $\lfloor \frac{n}{a_m} \rfloor$  is odd then  $\lfloor \frac{n}{a_m} \rfloor = 2l+1$ . Let  $n' = n - a_m$  then  $\lfloor \frac{n'}{a_m} \rfloor$  is even and by the above argument we can write  $n'$  as a sum of at most  $2(m-1)$  base elements, and hence in both situations  $n$  can be written as a sum of at most  $2m-1$  base elements.

If  $n = 2ma_m$ , it is clear that  $n$  can be written as a sum of  $m$  base elements, so now suppose that  $2ma_m < n \leq 4ma_m$ . We have already shown that  $4ma_m - n$  can be written as a sum of at most  $2m-1$  base elements; by the symmetry of  $A_k$  it follows that  $n$  can be written as a sum of at most  $2m$  base elements.

*Case (2):* Suppose first that  $n \leq (m + \frac{1}{2})(\hat{a}_m + a_m)$ . Write

$$n = \hat{q}\hat{a}_m + qa_m + r$$

where  $0 \leq r < a_m$  and  $|\hat{q} - q|$  is as small as possible. Note that in fact we can always find  $\hat{q}, q$  such that  $|\hat{q} - q| \leq 2$  by the following. If  $|\hat{q} - q| > 2$  then consider  $\lfloor \hat{a}_m/a_m \rfloor$ . If  $\lfloor \hat{a}_m/a_m \rfloor = 1$  and  $\hat{q} > q$  then we can rewrite  $n$  as

$$n = (\hat{q} - 1)\hat{a}_m + (q + 1)a_m + (r + x)$$

if  $x + r < a_m$ , and

$$n = (\hat{q} - 1)\hat{a}_m + (q + 2)a_m + (r + x - a_m)$$

otherwise. Alternatively if  $q > \hat{q}$  then we can rewrite  $n$  as

$$n = (\hat{q} + 1)\hat{a}_m + (q - 2)a_m + (r - x + a_m)$$

if  $r < x$ , or

$$n = (\hat{q} + 1)\hat{a}_m + (q - 1)a_m + (r - x)$$

otherwise. Similarly, if  $\lfloor \hat{a}_m/a_m \rfloor = 2$  then we can rewrite  $n$  as

$$n = (\hat{q} \mp 1)\hat{a}_m + (q \pm 2)a_m + (r \pm x \mp a_m)$$

or

$$n = (\hat{q} \mp 1)\hat{a}_m + (q \pm 3)a_m + (r \pm x \mp 2a_m).$$

Finally, if  $\lfloor \hat{a}_m/a_m \rfloor = 3$ , so  $\hat{a}_m = 3a_m$ , then we can rewrite  $n$  as

$$n = (\hat{q} \mp 1)\hat{a}_m + (q \pm 3)a_m + r.$$

Iterating this procedure we see that we can eventually arrive at coefficients for  $\hat{a}_m$  and  $a_m$  that differ by at most 2. Thus let

$$n = \hat{q}\hat{a}_m + qa_m + r$$

where  $0 \leq r < a_m$  and  $|\hat{q} - q| \leq 2$ .

Let

$$n' = \begin{cases} n - |\hat{q} - q|\hat{a}_m - \min(q, 1)M & \text{if } \hat{q} > q \\ n - |\hat{q} - q|\hat{a}_m - \min(\hat{q}, 1)M & \text{if } q > \hat{q} \\ n - \min(\hat{q}, 1)M & \text{if } \hat{q} = q. \end{cases}$$

Then  $\lfloor \frac{n'}{\hat{a}_m + a_m} \rfloor = \lfloor \frac{n'}{M} \rfloor \leq m - 1$  and  $n' - \lfloor \frac{n'}{M} \rfloor M < a_m$ . By Lemma 2.1 we can write  $n'$  as a sum of at most  $2(m - 1)$  base elements, and hence  $n$  can be written as a sum of at most  $2m + 1$  base elements.

If  $n > (m + \frac{1}{2})(\hat{a}_m + a_m)$ , we apply the symmetry of  $A_k$  as in Case (1), and we are done.  $\square$

### 3. AN UPPER BOUND

We now derive upper bounds on the  $a_i$  by describing conditions on symmetric sets  $A_k$  that force  $A_k$  not to be compact. For the remainder of this section let  $A_k = \{1 = a_1 < a_2 < \dots < a_{2m}\}$  be a symmetric base such that

(1) the  $a_i$  satisfy

$$\begin{aligned} a_1 &= 1 \\ a_i &\geq 8a_{i-1} \text{ for } 2 \leq i \leq m \\ a_{m+i} &= 2a_m - a_{m-i} \text{ for } 1 \leq i \leq m-1 \\ a_{2m} &= 2a_m \end{aligned}$$

or

(2) the  $a_i$  satisfy

$$\begin{aligned} a_1 &= 1 \\ a_i &\geq 8a_{i-1} \text{ for } 2 \leq i \leq m \\ \hat{a}_m &= a_m + x \text{ for } x \geq 7a_m \\ a_{m+i} &= \hat{a}_m + a_m - a_{m-i} \text{ for } 1 \leq i \leq m-1 \\ a_{2m} &= \hat{a}_m + a_m. \end{aligned}$$

**Theorem 3.1.**  $A_k$  is not compact.

*Proof. Case (1):* Suppose that  $A_k$  is compact. If  $n < a_m$ ,  $n$  must be written as a sum of base elements chosen from  $\{a_1, \dots, a_{m-1}\}$  using at most  $2m$  summands. There are  $\binom{3m-1}{m-1}$  such sums. Thus,  $a_m \leq \binom{3m-1}{m-1}$ .

Observe that

$$\binom{3m}{m-1} \leq \binom{3m-1}{i} \text{ for } m-1 \leq i \leq 2m.$$

Thus,

$$\binom{3m-1}{m-1} < \frac{\sum_i \binom{3m-1}{i}}{m+2} = \frac{2^{3m-1}}{m+2} \leq 2^{3m-3}$$

provided  $m \geq 2$ . Thus,  $a_m < 8^{m-1}$ , contradicting our choice of  $A_k$ .

*Case (2):* Similarly, suppose  $A_k$  is compact. If  $n < a_m$ , then  $n$  must be written as a sum of base elements from  $\{a_1, \dots, a_{m-1}\}$ , using at most  $2m + 1$  summands, so, by the same argument as before,  $a_m \leq \binom{3m}{m-1}$ .

Again, using the same argument, we find that

$$\binom{3m}{m-1} < \frac{2^{3m}}{m+3} \leq 2^{3m-3},$$

provided  $m \geq 5$ , so  $a_m < 8^{m-1}$ , contradicting our choice of  $A_k$ .  $\square$

*Remark 3.1.* Utilising the above ideas and Stirling's formula it is possible to improve  $8 \leq \frac{a_i}{a_{i-1}}$  to  $\frac{27}{4} \leq \frac{a_i}{a_{i-1}}$  for  $2 \leq i \leq m$ , however, we omit the lengthy calculations here.

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